"An Introduction to Hyperbolic Functions in Elementary Calculus"

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The customary introduction to hyperbolic functions mentions that the combinations

$$(1/2)(e^u + e^{-u})$$

and

$$(1/2)(e^u - e^{-u})$$

occur with sufficient frequency to warrant special names. These functions are analogous, respectively, to cos u and sin u. This article attempts to give a geometric justification for cosh and sinh, comparable to the functions of sin and cos as applied to the unit circle. This article describes a means of identifying

$$\cosh u = (1/2)(e^u + e^{-u})$$

and

$$\sinh u = (1/2)(e^u - e^{-u})$$

with the coordinates of a point (x, y) on a comparable "unit hyperbola," $x^2 - y^2 = 1$.

The functions cosh u and sinh u are the basic hyperbolic functions, and their relationship to the so-called unit hyperbola is our present concern. The basic hyperbolic functions should be presented to the student with some rationale. Suppose we start by considering the family of rectangular hyperbolas of the form $xy = k$, $k > 0$, and that portion of the curve that lies in the first quadrant. The value of k will be found for the one member of the set that is tangent to the unit circle in the first quadrant. In Figure 1, note that the line $y = x$ and the unit circle intersect at A, the point whose coordinates are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

When these values are substituted in $xy = k$, the value of k is $1/2$. It will be shown later, by a rotation of the axes, that this equation is indeed the unit hyperbola.

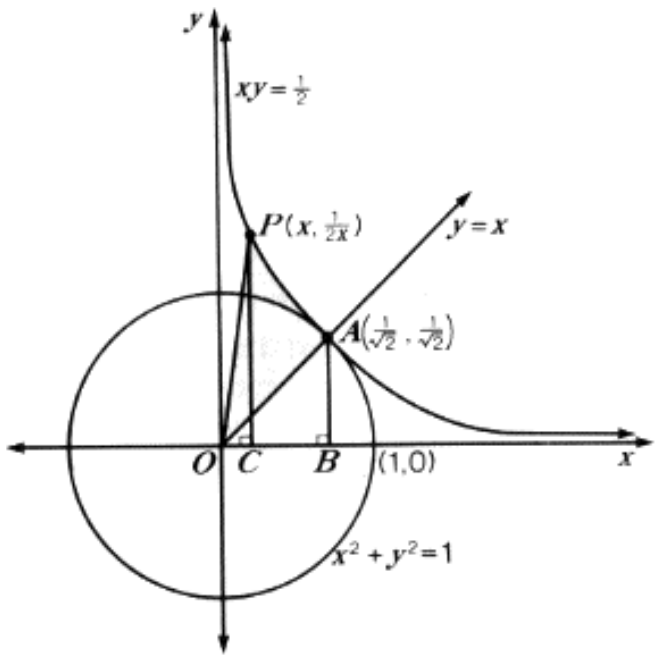

Figure 1

Let P be any point on $xy = 1/2$ with coordinates

$$P\left(x, \frac{1}{2x}\right),$$

where $x \leq 1/\sqrt{2}$. Draw $\overline{PC}$ and $\overline{AB}$ perpendicular to the x-axis and then draw $\overline{OP}$. Consider the area bounded by $\overline{OA}$, $\overline{OP}$, and the arc $\widehat{PA}$ of the hyperbola. Then we have, in terms of areas,

(1) $\quad OAP = OPC + PCBA - ABO,$

where ABO and OPC are right triangles and $PCBA$ is the area under the hyperbola from any point

$$P\left(x, \frac{1}{2x}\right) \text{ to } A\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Taking the areas in the same order as in equation (1), we have

(2) $$OAP = \frac{1}{2}(x)\left(\frac{1}{2x}\right) + \int_x^{1/\sqrt{2}} \frac{dx}{2x} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}.$$

Simplifying equation (2), we find that

$$OAP = \frac{1}{2} \log_e x \Big|_x^{1/\sqrt{2}},$$

so

(3) $$OAP = \frac{1}{2}\left(\log_e \frac{1}{\sqrt{2}} - \log_e x\right)$$

$$= \frac{1}{2}\log_e \frac{1}{x\sqrt{2}}.$$

Let the area bounded by $OAP = (1/2)u$ arbitrarily, so that

(4) $$\frac{1}{2}u = \frac{1}{2}\log_e \frac{1}{x\sqrt{2}}.$$

Equation (4) can be solved for x as follows:

$$u = \log_e \frac{1}{x\sqrt{2}}$$

$$e^u = \frac{1}{x\sqrt{2}}$$

(5) $$x = \frac{e^{-u}}{\sqrt{2}}.$$

Substituting the value of x in equation (5) into the equation $xy = (1/2)$, we have

(6) $$y = \frac{e^u}{\sqrt{2}}.$$

Thus, the coordinates of any point on the hyperbola $xy = (1/2)$ can be represented by

(7) $$\left(\frac{e^{-u}}{\sqrt{2}}, \frac{e^u}{\sqrt{2}}\right),$$

where $(1/2)u$ is the area of OAP, as shown in Figure 2.

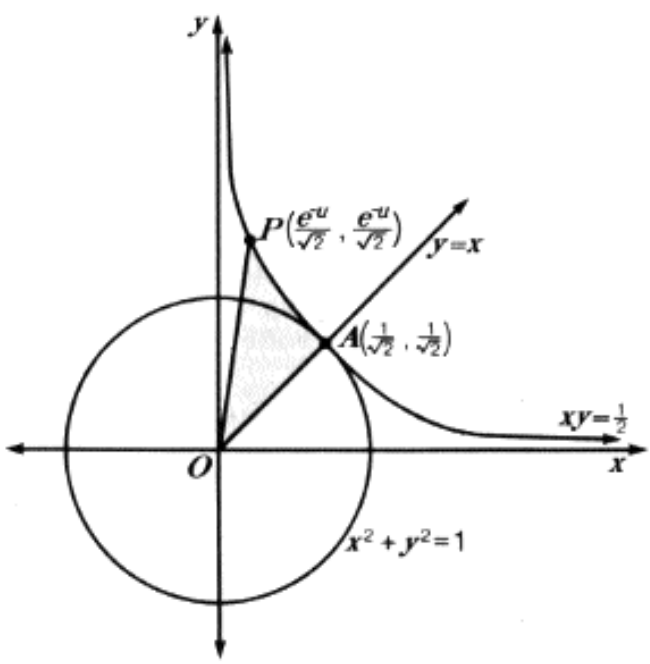

Figure 2

Apply the rotational formulas

$$x = x' \cos\theta - y' \sin\theta$$

and

$$y = x' \sin\theta + y' \cos\theta$$

and rotate the axes 45° counterclockwise. Note that the area *OAP* remains unaltered. After the primes are dropped, the rotational formulas become

$$\frac{e^{-u}}{\sqrt{2}} = x\left(\frac{1}{\sqrt{2}}\right) - y\left(\frac{1}{\sqrt{2}}\right) \tag{8}$$

and

$$\frac{e^{u}}{\sqrt{2}} = x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right). \tag{9}$$

Simplifying equations (8) and (9) gives

$$e^{-u} = x - y \tag{10}$$

and

$$e^{u} = x + y. \tag{11}$$

By solving equations (10) and (11) for x and y, we have

$$x = \frac{e^{u} + e^{-u}}{2} \tag{12}$$

and

$$y = \frac{e^{u} - e^{-u}}{2}. \tag{13}$$

Equations (12) and (13) yield

$$x^2 - y^2 = 1. \tag{14}$$

Interpreted simply, we have shown that any point on the hyperbola $x^2 - y^2 = 1$ has the coordinates given in equations (12) and (13), where $(1/2)u$ is the area shown in Figure 3. The values of x and y are cumbersome, and the following statements will define two of the hyperbolic functions:

$$\sinh u = (1/2)(e^{u} - e^{-u}) \tag{15}$$

and

$$\cosh u = (1/2)(e^{u} + e^{-u}). \tag{16}$$

The third member of the primary hyperbolic functions is defined as the hyperbolic tangent in the following manner:

$$\tanh u = \frac{\sinh u}{\cosh u} \tag{17}$$

or, in terms of e,

(18) $$\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

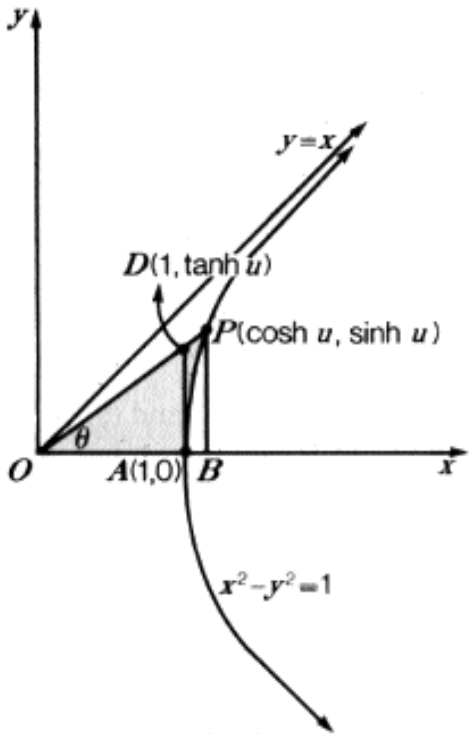

Figure 3

A geometric interpretation of tanh u can be obtained from Figure 3. If $\overline{AD}$ and $\overline{PB}$ are drawn perpendicular to the x-axis, then OAD and OBP are similar triangles with proportional sides. Therefore,

$$\frac{AD}{OA} = \frac{BP}{OB}.$$

Substituting the values given in Figure 3, we see that $AD = \tanh u$ and that point A has the coordinates $(1, \tanh u)$. Also, note that if angle DOA is designated as θ, then for all $\theta < 45°$,

(19) $$\tanh u = \tan\theta.$$

Equation (19) is a link between the circular and the hyperbolic functions.

Questions as to the placement of this topic within the mathematical curriculum and the depth of the knowledge sought are best answered by the individual instructor. It has been my experience that the best results are obtained if this lesson is taught when the term *hyperbolic function* is introduced. The development is similar to the manner in which sine, cosine, and tangent are initially defined in terms of a right triangle. For an Advanced Placement calculus class, the topic might be used as the basis for a research paper or special assignment.