The problem of finding the volumes of solid figures can be used by calculus teachers to instill in their students the very useful habit of checking the plausibility of their answers against previously made intuitive estimates. Furthermore, the process used by students to arrive at their estimated answers might help them to develop the necessary insights for exact mathematical solutions.

The purpose of this article is to offer some examples of interesting solid figures that students can use to “exercise” their estimating skills. What makes the figures considered here so interesting is that although they are bounded by surfaces that are portions of cylinders, their volumes can be expressed as rational numbers. Therefore, a student’s rational estimate might actually turn out to be the exact answer.

**Solid Figures Formed by the Intersection of Two Right Circular Cylinders**

Geometric problems offer especially good subject matter for emphasizing the estimating aspect of problem solving. In this regard, one of my favorite calculus problems in the unit on applications of integration goes as follows:

Find the volume of the solid figure formed by the intersection of two right circular cylinders of radius \( r \) whose axes intersect at right angles (fig.1).
Fig. 1. What is the volume of the solid figure formed by the intersection of two right circular cylinders?

At first, many students think that the intersection is a sphere. To help the students visualize the actual shape of the solid, I ask them to make a cardboard model of the intersection (with $r = 2$ in.) and to estimate its volume before using calculus to find the exact answer. A sketch of the top half of the solid figure is given in figure 2.

Fig. 2. The top half of the solid figure formed in figure 1 when $r = 2$.

Stannard (1979) called this solid a “birdcage.” It is important to notice that cross sections of the birdcage perpendicular to the $y$-axis are squares. This fact surprises many students; it is contrary to their intuitions.

Most students estimate the volume of the birdcage by assuming that it is “roughly” the same as that of a sphere of radius 2 inches. Hence, the volume is approximately

$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (2)^3 = 33.5 \text{ in.}^3.$$
One of my students, Jennifer Johnson, estimated the volume to be the average of the volumes of a cube with an edge of 4 inches and the inscribed octahedron (see fig. 3). As it turns out, this estimate is precisely the answer.

Fig. 3. The volume of the birdcage (shaded in the top figure) is the average of the volumes of the circumscribed cube and the inscribed octahedron (shaded in the bottom figure).

\[
V = \frac{e^3}{2} + \frac{2((1/3)Bh)}{2} = \frac{64}{2} + \frac{(64/3)}{2} = \frac{42}{3} \text{ in}^3
\]

It may be that Jennifer intuitively recognized that the ratio of the volume of a birdcage to the volume of its circumscribed cube is the same as the ratio of the volume of a sphere to the volume of its circumscribed cylinder. Figure 4 shows the top halves of these solid figures.
Fig. 4. The volume of a birdcage is two-thirds the volume of its circumscribed cube. Similarly, the volume of a sphere is two-thirds the volume of its circumscribed cylinder.

In general, the volume of a sphere of radius $r$ is two-thirds the volume of the circumscribed cylinder.

$$V = \frac{2}{3} \pi r^2 h = \frac{2}{3} \pi r^2 (2r) = \frac{4}{3} \pi r^3$$

Similarly, the volume of a birdcage is two-thirds the volume of the circumscribed cube.

$$V = \frac{2}{3} (2r)^3 = \frac{16}{3} r^3$$

If $r = 2$ inches,

$$V = \frac{16}{3} (2)^3 = 42 \frac{2}{3} \text{ in.}^3.$$  

Since cross sections of the birdcage in figure 2 taken perpendicular to the $y$-axis are squares, we can use calculus (volumes of solids with known cross sections) to compute the exact volume:

$$V = \int_{-2}^{2} (2 \sqrt{4 - y^2})^2 \, dy = 42 \frac{2}{3} \text{ in.}^3$$

The problem of finding the volume of the solid formed by the intersection of two right cylinders of radius $r$ can be generalized by having the axes of the cylinders intersect at acute angle $\theta$ (see fig. 5).
Here, cross sections perpendicular to the $y$-axis are rhombuses. In figure 5, rhombus $ABCD$ is a cross section whose area is

$$A = bh$$

$$= \frac{2\sqrt{r^2 - y^2}}{\sin \theta} \cdot 2\sqrt{r^2 - y^2}$$

$$= \frac{4}{\sin \theta} (r^2 - y^2).$$

**Fig. 5.** Cross sections of the solid figure formed by the intersection of these right cylinders are rhombuses.

If we estimate the volume of the birdcage by taking two-thirds the volume of the circumscribed prism, we obtain

$$V = \frac{2}{3} Bh = \frac{2}{3} \cdot \frac{2r}{\sin \theta} \cdot 2r \cdot 2r = \frac{16}{3 \sin \theta} r^3.$$

Using calculus to compute the exact volume, we have

$$V = \int_{-r}^{r} \frac{4}{\sin \theta} (r^2 - y^2) dy = \frac{16}{3 \sin \theta} r^3.$$

Again, the estimate turns out to be the exact volume, and the volume is rational when $r$ and $\sin \theta$ are rational numbers.
Fig. 6. The wedge formed by the intersection of a plane through the diameter of a base of a right circular cylinder might have a rational volume.

Fig. 7. The volume of the wedge formed in figure 6 is one-third the volume of the circumscribed rectangular solid.
Fig. 8. The volume of this wedge is not a rational number.

Wedges

The wedge formed by the intersection of a plane through a diameter of the base of a right circular cylinder of radius \( r \) and making an angle \( \theta \) with the base (fig. 6) is another figure that might have a rational volume. Here, cross sections perpendicular to the \( x \)-axis are right triangles, whereas cross sections perpendicular to the \( y \)-axis are rectangles.

From using cross sections perpendicular to the \( x \)-axis (see fig. 6), the area of triangle \( ABC \) is given by

\[
A = \frac{1}{2}bh = \frac{1}{2} \sqrt{r^2 - x^2} \cdot \sqrt{r^2 - x^2} \tan \theta
\]

\[
= \frac{\tan \theta}{2} (r^2 - x^2),
\]

and the volume of the wedge is

\[
V = \int_{-r}^{r} \frac{\tan \theta}{2} (r^2 - x^2)dx = \frac{2}{3} r^3 \tan \theta.
\]

Therefore, when \( r \) and \( \tan \theta \) are rational numbers the volume of the wedge is a rational number. The estimate of the volume found by taking one-third the volume of the circumscribed rectangular solid (see fig. 7) turns out to be the exact answer. Notice that the ratio of the volume of this wedge to the volume of the circumscribed rectangular solid is the same as the ratio of the volume of a cone to the volume of its circumscribed cylinder.
It is of interest to note that if the cutting plane does not pass through a diameter of the base of the cylinder, the volume of the wedge that is formed is not a rational number. For example, the volume of the wedge formed by the intersection of the plane and the cylinder in figure 8 is

\[ V = \int_{-1}^{2} 2\sqrt{4 - y^2} (1 + y)dy \]

\[ = \frac{8}{3} \pi + 3\sqrt{3} \approx 13.6. \]

Here, rectangular cross sections perpendicular to the $y$-axis are used to simplify the integration. Notice that cross sections perpendicular to the $x$-axis are triangles for $|x| \leq \sqrt{3}$ and are trapezoids for $\sqrt{3} < |x| < 2$. The estimate of the volume of the wedge found by taking one-third the circumscribed rectangular solid is

\[ V = \frac{1}{3} l \cdot w \cdot h = \frac{1}{3} (4)(3)(3) = 12, \]

a slight underestimate.

**Summary**

The solid figures considered in this article are figures whose formation can be clearly conceived by students because the solids are formed by the intersections of familiar figures. Instructors can help their students to visualize the figures by asking them actually to construct cardboard models. However, the problem of computing their volumes presents conceptual, artistic, and mathematical challenges. Furthermore, the fact that the volumes can sometimes be expressed by rational numbers offers the instructor the opportunity to reinforce the idea of “estimating one’s answers” and to reflect on the virtues and pitfalls of intuition.

**Reference**