Exploding the Ellipse

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Readers are advised to proceed with caution. Those with a weak heart may wish to consult a physician first. What we are about to do is explode an ellipse. This risky business is not often undertaken by the professional mathematician, whose polytechnic endeavors are usually limited to encounters with administrators.

Ellipses of the standard form of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > b$, are not suitable for exploding because they just move out of view as they explode. Hence, before the ellipse explodes, we must secure it in the neighborhood of the origin by translating the left vertex to the origin and anchoring the left focus to a point on the x-axis. Then a portion of the ellipse will always remain in view.

At this point, recall that the ellipse is characterized by two focal points that lie on the horizontal axis; their distance from the center of the ellipse is c, where $c^2 = a^2 - b^2$. **Figure 1** shows the graph of an ellipse in standard form. The equation of the ellipse after translation is

$$\frac{(x - a)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

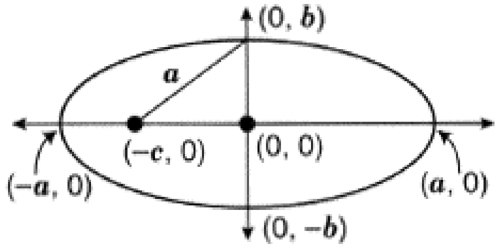

Figure 1
An ellipse in poor form for exploding

The left vertex, which has been translated to the origin, and the left focus, which has been anchored to the point $(k, 0)$, where $k = a - c$, are fixed in place. Throughout the entire procedure, k will remain constant but a, b, and c will undergo drastic change; therefore, precautions may be necessary. **Figure 2** shows the translated ellipse.

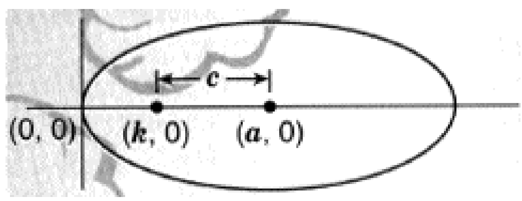

Figure 2
An ellipse in a form that is ready for exploding

Because k remains unchanged, we change the form of the equation of the ellipse so that k appears in the equation, whereas b, which changes, does not appear.

Since $k = a - c$ and $b^2 = a^2 - c^2 = (a - c)(a + c) = k(a + c)$, the equation of the ellipse can be written as

$$\frac{(x - a)^2}{a^2} + \frac{y^2}{k(a + c)} = 1.$$

The process of squaring and simplifying yields

$$(1) \qquad \frac{x^2}{a} - 2x + \frac{y^2}{k\left(1 + \frac{c}{a}\right)} = 0.$$

Because $c = a - k$, we have

$$\frac{x^2}{a} - 2x + \frac{y^2}{k\left(1 + \frac{a - k}{a}\right)} = 0,$$

or

$$(2) \qquad \frac{x^2}{a} - 2x + \frac{y^2}{k\left(2 - \frac{k}{a}\right)} = 0.$$

Alas, the beauty and simplicity of the standard form of the equation of the eillipse are gone. We wonder: Are we doing the right thing? But it is too late to turn back.

We are now ready to proceed with the expansion and ultimate explosion of the ellipse. At this point, readers should consider putting on dark glasses, because things may become illuminating. Also, remove or tie down such loose objects as pencils and calculators.

Okay, here goes. We let a expand unchecked. It picks up speed as it races to infinity, taking with it b and c and the center of the ellipse. (See **fig. 3.**)

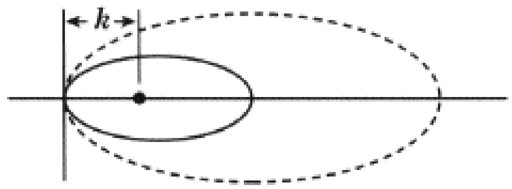

Figure 3
Ellipse during expansion

Recall that a moves toward infinity, whereas k remains fixed.

Boom!

Wow, did you see that? Are you okay?

While, concerned for our personal safety, we covered our eyes, a approached infinity with

$$\lim_{a\to\infty}\left(\frac{x^2}{a}\right) = 0$$

and

$$\lim_{a\to\infty}\left[\frac{y^2}{k\left(2-\frac{k}{a}\right)}\right] = \frac{y^2}{2k}.$$

In an instant, all that we learned about an ellipse, that is, equation (2), has been taken to the limit.

$$\lim_{a\to\infty}\left\{\frac{x^2}{a} - 2x + \frac{y^2}{k\left(2-\frac{k}{a}\right)}\right\} = \lim_{a\to\infty}\{0\}$$

$$\left\{\lim_{a\to\infty}\left(\frac{x^2}{a}\right)\right\} + \left\{\lim_{a\to\infty}(-2x)\right\} + \left\{\lim_{a\to\infty}\frac{y^2}{k\left(2-\frac{k}{a}\right)}\right\} = 0$$

$$-2x + \frac{y^2}{2k} = 0$$

$$y^2 = 4kx$$

The ellipse is gone, but look at the beautiful parabola that results (**fig. 4**).

Please advise your students that this exercise should be done only by a professional mathematician and is not something that they should try at home.

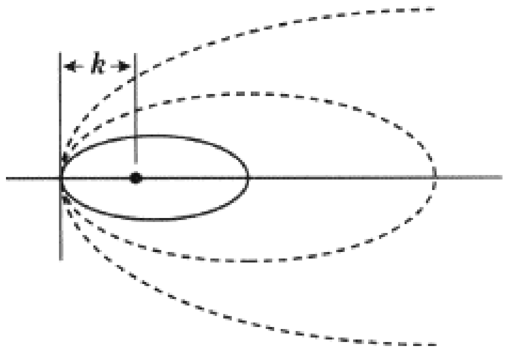

Figure 4
Ellipse during and after expansion

In retrospect, we recognize that the family of ellipses defined by fixed k are all contained within the parabola $y^2 = 4kx$, which is the limiting value of equation (2) as a goes to infinity. The value k, originally the distance from the left focus of the ellipse to its left vertex, is now the distance from the focus of the parabola to its vertex. We now have precisely the standard form of the equation of the parabola. How fortunate.

EPILOGUE

As the center of the ellipse went to infinity, so did the second, right, focus. The sum of the distances from a point on the ellipse to the foci is constant. We can ask whether the sum of the distances from a point on the parabola, that is, the exploded ellipse, to the foci is also a constant. We draw a ray from a point on the parabola to the second, or right, focus, labeled G, which is at infinity. (See **fig. 5.**) This ray is drawn parallel to the horizontal axis and contains the second focus, which is on the horizontal axis, because all parallel lines meet at infinity.

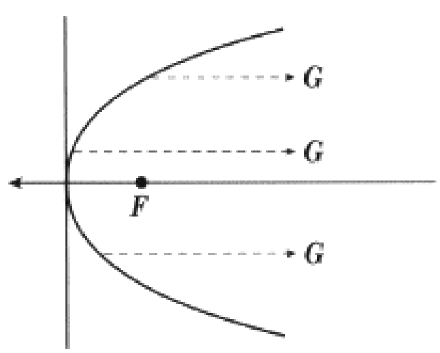

Figure 5
Parabola with rays to right focus

At this point, we remind ourselves that a parabola can be defined as the loci of all points equidistant from a specified point—the focus, *F*, and a line—the directrix, as shown in **figure 6.**

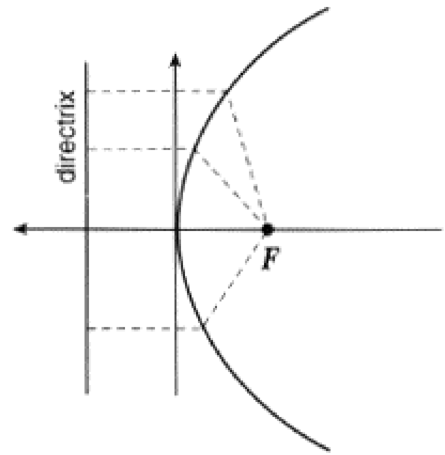

Figure 6
Parabola with focus, *F*, and directrix

We extend each ray in the negative *x* direction by a length equal to the distance from that point to the focus of the parabola—the initial focus of the exploded ellipse. Each ray, so extended, ends at the directrix. Consequently, the endpoints of these extended rays are collinear and fall on a line perpendicular to the horizontal axis, as shown in **figure 7.**

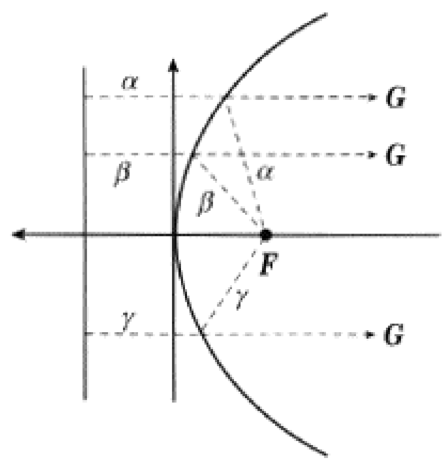

Figure 7
Parabola with extended rays

The length of each ray before its extension is ∞; by adding a finite extension, it remains infinite. That is, $\infty + x = \infty + y$ is true whether or not $x = y$. Therefore, exploring the length of these extended rays algebraically has no merit. Geometrically, the situation is different. Looking at **figure 7,** can we not say that since each extended ray emanates from the same vertical line, all are of equal length? We recall that each extension made to the directrix is equal to the length of the line segment drawn from the point from which the extension was made to the focus of the parabola. Consequently, using graphical arguments, we can say that the sum of the distances from a point on the parabola to its “two” foci—the finite one, F, and the one now at infinity, G—are equal for all such points. After all, this sum, represented graphically, equals the length of the extended rays.

So is a parabola merely an exploded ellipse? And the point at infinity just the other focus far away? To paraphrase the Welch metaphysical poet Henry Vaughan,

> . . . Had I seen infinity the other night
> Would it have been a point of endless light?[1]

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[1]The actual lines from Henry Vaughan’s *Silex Scintillans* are as follows: “I saw eternity the other night/Like a great Ring of pure and endless light.”

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